BASIC CONCEPTS IN PHYSICS

DERIVATIONS OF EQUATIONS IN THE TEXT
BY BILL DANIEL

CHAPTER 1

(1.27) BY LETTING $\theta \to \theta + d\theta$, USING TRIG IDENTITIES FOR $\sin(\alpha+\beta)$ AND $\cos(\alpha+\beta)$ AND RECOGNIZING THAT $d\vec{\theta} = d\theta\,\hat{k}$, IT IS POSSIBLE WITH SOME MESSY ALGEBRA TO PROVE (1.27) FROM (1.10), BUT I THINK THIS OBSCURES WHAT'S REALLY GOING ON. WHAT WE ARE TRYING TO DO IS TO EXPRESS THE RATE OF CHANGE OF AN ARBITRARY VECTOR QUANTITY, $\vec{r}(t)$, AS MEASURED IN AN INERTIAL FRAME, S, IN TERMS OF ITS RATE OF CHANGE IN A ROTATING FRAME, S'.

FIRST, WRITE $\vec{r}$ IN ITS CARTESIAN COORDINATES IN S:

$$\vec{r} = \sum_{i=1}^{3} r_i \hat{e}_i \qquad [1.1]$$

THE UNIT VECTORS $\hat{e}_i$ ARE FIXED IN S, BUT CHANGING (ROTATING) IN S'. DIFFERENTIATING W.R.T. TIME GIVES:

$$\frac{d\vec{r}}{dt} = \sum_i \frac{dr_i}{dt}\hat{e}_i \qquad [1.2]$$

IN FRAME S. BUT, IN S' WITH CHANGING $\hat{e}_i$, WE MUST WRITE

$$\frac{d\vec{r}}{dt} = \sum_i \frac{dr_i'}{dt}\hat{e}_i + \sum_i r_i\left(\frac{d\hat{e}_i'}{dt}\right) \qquad [1.3]$$

NOW, (1.26) TELLS US HOW TO WRITE THE SECOND TERM ON THE RIGHT. EQN (1.26) GAVE US THE INSTANTANEOUS LINEAR VELOCITY $\left(\frac{d\vec{A}}{dt}\right)$ OF A POINT $(\vec{A})$ ROTATING ABOUT AN AXIS THROUGH THE ORIGIN WITH ANGULAR VELOCITY $\vec{\omega}$. THE $\hat{e}_i$ UNIT VECTORS ARE ROTATING WITH ANGULAR VELOCITY $\vec{\omega}$, SO:

$$\sum_i r_i \frac{d\hat{e}_i'}{dt} = \vec{\omega} \times \vec{r} \qquad [1.4]$$

SO, COMBINING [1.2] AND [1.4] GIVES (1.27). (NOTE THAT I AM USING SQUARE BRACKETS, [ ], TO INDICATE EQUATIONS IN THESE NOTES AND PARANTHESES ( ) TO INDICATE THOSE IN THE TEXT.)

(1.28) THIS IS JUST ANOTHER APPLICATION OF THE ABOVE DIFFERENTIATION. DIFFERENTIATING (1.27) GIVES:

$$\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}' + \vec{\omega} \times \vec{r}\right) \qquad [1.5]$$

NOTE THAT BOTH OF THE OUTER DERIVATIVES ARE W.R.T. THE INERTIAL FRAME, S, BUT THE INNER DERIVATIVE ON THE RIGHT IS W.R.T. THE ROTATING FRAME, S'. WE WANT TO WRITE THE SECOND DERIVATE ON THE LEFT (W.R.T THE INERTIAL FRAME) IN TERMS ON THE RIGHT <u>ONLY</u> W.R.T. THE ROTATING FRAME. TO DO THIS WE ONCE AGAIN USE (1.27) TO WRITE

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}'\left[\frac{dr}{dt}' + \vec{\omega} \times \vec{r}\right] + \vec{\omega} \times \left[\frac{d\vec{r}}{dt}' + \vec{\omega} \times \vec{r}\right] \qquad [1.6]$$

WHERE I'M USING $\frac{d'}{dt}$ TO INDICATE DIFFERENTIATION W.R.T. THE ROTATING FRAME.

NOTING THAT $\vec{\omega}$ IS CONSTANT, SO $\frac{d'\vec{\omega}}{dt} = 0$, AND

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}\,'}{dt^2} + \vec{\omega}\times\frac{d\vec{r}\,'}{dt} + \vec{\omega}\times\frac{d\vec{r}\,'}{dt} + \vec{\omega}\times\vec{\omega}\times\vec{r}$$

OR

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}\,'}{dt^2} + 2\vec{\omega}\times\frac{d\vec{r}\,'}{dt} + \vec{\omega}\times\vec{\omega}\times\vec{r} \qquad (1.28)$$

(1.33) IN 1 DIMENSION, $q_i = q_1 = x$ AND $\dot{q}_i = \dot{x} = v$.

SINCE $L = \frac{1}{2}mv^2$, $\frac{\partial L}{\partial v} = mv$ AND $\frac{\partial L}{\partial x} = 0$, SO BY EULER-LAGRANGE,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}(mv) - 0 = m\frac{dv}{dt} = 0 \qquad (1.33)$$

(1.36) USING THE EQUATION FOR A PARTICLE OF MASS m MOVING IN A POTENTIAL $V(x)$ GIVEN IN THE BOOK BETWEEN (1.30) AND (1.31) AND (1.19) FOR GRAVITATIONAL POTENTIAL,

$$L = \frac{1}{2}mv^2 + \frac{GMm}{r}$$

THIS BECOMES (1.36) WHEN WE USE (1.35) AND RECOGNIZE THAT $v^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + r^2\dot{\theta}^2$.

(1.37) $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}^2$; $\frac{\partial L}{\partial \theta} = 0$, SO E-L GIVES (1.37).

(4)

RECOGNIZE THAT L IN (1.36) IS $L = T - V$, SO

(1.38) $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$ AND $V = -\frac{GMm}{r}$, SO $E = T + V$

AND AND $\dot{\theta} = C/mr^2$ GIVES (1.38). THE FOLLOWING

FOLLOWING EQUATION FOR $\dot{r}$ IS JUST AN ALGEBRAIC REARRANGEMENT.

TO GET THE NEXT EQN FOR $d\theta$, USE

$$\dot{r} = dr/dt = \sqrt{\tfrac{2}{m}[\,]} \longrightarrow dt = dr/\sqrt{\tfrac{2}{m}[\,]} \qquad [1.7]$$

WHERE [ ] IS THE BRACKET IN THE EQN FOR $\dot{r}$ FOLLOWING (1-38)

$$\dot{\theta} = d\theta/dt \longrightarrow d\theta = \dot{\theta}dt = C\,dt/mr^2$$

SUBSTITUTING FOR dt FROM [1.7]

$$d\theta = C\,dr/mr^2\sqrt{\tfrac{2}{m}[\,]} = \frac{C\,dr/r^2}{\sqrt{2m[\,]}}$$

(1.39) REWRITE $d\theta$ AS:

$$d\theta = C\left[\frac{dr}{r\sqrt{a + br + cr^2}}\right] \text{ WHERE } a = -C^2;\ b = 2GMm^2;\ c = 2mE$$

USING INTEGRAL TABLES, SINCE $a < 0$,

$$\int d\theta = \theta = C\left[\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{br + 2a}{r\sqrt{b^2 - 4ac}}\right)\right]$$

$$= \sin^{-1}\left(\frac{2GMm^2 - 2C^2}{r\sqrt{4G^2M^2m^4 + 8C^2mE}}\right)$$

$$= \sin^{-1}\left(\frac{C/r - GMm^2/C}{\sqrt{2mE + G^2M^2m^4/C^2}}\right)$$

AS OPPOSED TO THE $\cos^{-1}$ IN (1.39). NOT SURE WHY THE DIFFERENCE, BUT EITHER WILL GIVE THE CONIC EQN IN (1.40).

(1,40) JUST MESSY ALGEBRA:

$$\cos\theta = \frac{(1/r - 1/d)}{\sqrt{\frac{2mE}{G^2} + 1/d^2}} = \frac{\left(\frac{1}{r} - \frac{1}{d}\right)}{\frac{1}{d}\sqrt{1 + \frac{2mEd^2}{G^2}}}$$

$$= \frac{\frac{d}{r} - 1}{\sqrt{1 + 2EG^2/G^2M^2m^3}} = \frac{\frac{d}{r} - 1}{\varepsilon}$$

$$1 + \varepsilon\cos\theta = \frac{d}{r} \longrightarrow r = \frac{d}{1 + \varepsilon\cos\theta} \quad (1,40)$$

§ 1,9 SINCE $L = \sum_{i=1}^{N} L(\dot{q}_i, q_i, t)$, WE FORM ITS TOTAL DIFFERENTIAL BY THE RULE:

$$dr(x,y,z) = \frac{\partial r}{\partial x} \cdot dy \cdot dz + dx \cdot \frac{\partial r}{\partial y} \cdot dz + dx \cdot dy \cdot \frac{\partial r}{\partial z}.$$

(1.41) SINCE THE CANONICAL MOMENTA ARE

$$p_i = \partial L / \partial \dot{q}_i$$

THE SECOND TERM IN (1.41) COMES FROM A SUBSTITUTION. THE FIRST TERM IS FOUND FROM THE EULER-LAGRANGE EQN, (1.32):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \dot{p}_i = \frac{\partial L}{\partial q_i}.$$

THE EQN. IN THE TEXT FOLLOWING (1.41) IS DIFFERENTIATION OF A PRODUCT: $d(p_i \dot{q}_i) = p_i d\dot{q}_i + \dot{q}_i dp_i$.

(1.42) SINCE $H = \sum_i p_i \dot{q}_i - L$, AND $dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt$

$$dH = \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - dL$$

$$= \cancel{\sum_i p_i d\dot{q}_i} + \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \cancel{\sum_i p_i d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

$$dH = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt \qquad (1.42)$$

(1.43) BECAUSE L DOES NOT DEPEND EXPLICITLY ON p_i,

$$\frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\sum_i p_i \dot{q}_i - L\right) = \dot{q}_i$$

$$\frac{\partial H}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\sum_i p_i \dot{q}_i - L\right) = -\frac{\partial L}{\partial q_i} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$= -\frac{d}{dt} p_i = -\dot{p}_i$$

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}\left(\sum_i p_i \dot{q}_i - L\right) = -\frac{\partial L}{\partial t}$$

p. 44 $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$ AND $\dot{p} = -\frac{\partial H}{\partial q} = -kq$

$$\ddot{q} = \frac{\dot{p}}{m} = -\frac{k}{m} q$$

THE SIMPLEST SECOND ORDER O.D.E., $\ddot{x} = -ax$, HAS THE SOLUTION $x = A \sin\sqrt{a}t + B\cos\sqrt{a}t = C\cos(\sqrt{a}t + \phi)$ WITH A AND B, OR C AND ϕ ARBITRARY.

(1.44) FOLLOWS FROM THE DEF'N OF THE TOTAL DERIVATIVE.

(1.45) SINCE $\dot{q}_i = \frac{\partial H}{\partial p_i}$ AND $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, (1.44) BECOMES

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\sum_i \left(\frac{\partial f}{\partial q_i} \cdot \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \cdot \frac{\partial H}{\partial q_i} \right)}_{\{H,\, f\}}$$

(1.46) SINCE $p^2 = p_x^2 + p_y^2 + p_z^2$

$$\frac{\partial H}{\partial x} = \frac{\partial V}{\partial x} \qquad \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \frac{\partial L}{\partial x} = p_y \qquad \frac{\partial L}{\partial p_x} = -y$$

$$\frac{\partial H}{\partial y} = \frac{\partial V}{\partial y} \qquad \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \qquad \frac{\partial L}{\partial y} = -p_x \qquad \frac{\partial L}{\partial p_y} = x$$

$$\frac{\partial H}{\partial z} = \frac{\partial V}{\partial z} \qquad \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \frac{\partial L}{\partial z} = 0 \qquad \frac{\partial L}{\partial z} = 0$$

$$\{H, L_z\} = \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial L}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial L}{\partial p_i} \right) = \frac{p_x}{m} \cdot p_y - \frac{\partial V}{\partial x}(-y) + \frac{p_y}{m}(-p_x) - \frac{\partial V}{\partial y}(x)$$

$$= y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y} \qquad (1.46)$$

CHAPTER 2

(2.5) THERE IS AN ERROR IN THE CALCULATON OF THE PROBABILITY THAT n LIES IN THE $\frac{N}{2} \pm \frac{N}{4}$ RANGE WHEN $N = 12$. IT SHOULD BE

$$P(3 \le n \le 9) = \frac{924 + 2(792) + 2(495) + 2(440)}{2^{12}}$$

$$= \frac{3938}{4096} = 0.961.$$

NOT $\frac{987}{1024} = 0.964.$

ALSO (2.5) APPEARS TO BE IN ERROR. CLEARLY IT'S NOT TRUE FOR SYMMETRIC DISTRIBUTIONS WHERE $n = N/2$, SINCE THE L.H.S. = 0. THE FIRST UNSYMMETRIC CASE FOR $N = 6$, THE CASE OF $n = 4, n' = 2$, GIVES:

$$\frac{n - N/2}{N} = \frac{4 - 6/2}{6} = \frac{1}{6}$$

THERE IS NO INTEGER q FOR WHICH $2^q = 6$.

I HAVE NOT WORKED THIS OUT, BUT I THINK IT CAN BE SHOWN THAT FOR N AN EVEN INTEGER:

$$\frac{n - \frac{N}{2}}{N} > 1/2^{N/2} \qquad [2.1]$$

FOR $\frac{N}{2} < n < N$. THIS IS THE CLOSEST I CAN GET TO (2.5). I'M ATTACHING 3 PAGES FROM D. SCHROEDER, THERMAL PHYSICS THAT WILL GET YOU (ALMOST) TO (2.9), EQN (2.6) IN SCHROEDER.

(2.10) LET $n' \to 0$ $(n \to N)$ AND

$$P_{N,0} = \frac{N!}{0!\,N!}\left(\frac{1}{2}\right)^N\left(\frac{1}{2}\right)^0 = 2^{-N}$$

(2.13) GIVEN DEF'N (2.12) WITH N ALLOWED STATES EACH WITH AN EQUAL PROBABILITY $P_i = 1/N$ FOR $i = 1, 2, \cdots, N$.

$$S = -k \sum_i P_i \ln P_i = -k N \cdot \left(\frac{1}{N} \ln \frac{1}{N}\right).$$

RECALL $\ln\left(\frac{1}{N}\right) = \ln N^{-1} = -\ln N$, SO

$$S = -kN \cdot \left(\frac{1}{N} \cdot (-\ln N)\right) = k \ln N. \qquad [2.2]$$

(2.17) WITH S A FUNCTION OF δE_1 AND δE_2 THIS IS THE STATEMENT OF THE TOTAL DIFFERENTIAL OF S.

(2.20) I THINK THE AUTHORS OWE THE READER A BETTER EXPLANATION OF WHERE THIS EXPRESSION COMES FROM, SO I'LL TRY TO GIVE ONE:

IN AN ISOLATED SYSTEM, ALL MICROSTATES ARE EQUALLY PROBABLE. IN A CANONICAL ENSEMBLE, THOUGH, THE SYSTEM IS NOT ISOLATED, IT "EXCHANGES ENERGY WITH A LARGE 'HEAT BATH'." THE COMBINATION OF THE SYSTEM AND THE "BATH", IS, HOWEVER, ISOLATED, SO ITS COMBINED STATES HAVE THE SAME PROBABILITY. IF THERE ARE N_i BATH MICRO STATES ASSOCIATED WITH STATE S_i OF THE SYSTEM, THEN THE PROBABILITIES

OF SYSTEM STATES S_1 AND S_2 ARE IN THE SAME RATIO AS THE NUMBER OF BATH STATES:

$$\frac{P(s_1)}{P(s_2)} = \frac{N_1}{N_2} \qquad [2.3]$$

SINCE THE ENTROPY OF SYSTEM PLUS BATH ASSOCIATED WITH STATE 1 IS JUST $S_1 = k \ln N_1$, THE PROBABILITES ARE ALSO IN THE RATIO OF THE EXPONENTIALS OF THE BATH ENTROPIES:

$$\frac{P(s_1)}{P(s_2)} = \frac{e^{S_1/k}}{e^{S_2/k}} = e^{(S_1-S_2)/k} \qquad [2.4]$$

IF THE SYSTEM (MAY BE ONE ATOM) IS SMALL COMPARED TO THE RESERVOIR, WE CAN WRITE:

$$S_1 - S_2 = \frac{1}{T}[U_1 - U_2]_{BATH} = -\frac{1}{T}[E_1 - E_2]_{SYSTEM}$$

SO $\dfrac{P(s_1)}{P(s_2)} = \dfrac{e^{-E_1/kT}}{e^{-E_2/kT}}$. EACH EXPONENTIAL (CALLED A BOLTZMAN FACTOR) IS A FUNCTION OF THE ENERGY OF A PARTICULAR MICROSTATE & THE TEMPERATURE. SO FOR ANY PARTICULAR STATE, S, OF THE SYSTEM,

$$P(s) = \frac{1}{Z} e^{-E(s)/kT} \qquad [2.5]$$

WHERE $1/Z$ IS A PROPORTIONALITY CONSTANT CALLED THE PARTITION FUNCTION. IF WE NOW ALLOW THE STATES, S, TO BECOME CONTINUOUS, THE PROBABILITY BECOMES A PROBABILITY DENSITY, ρ, SO

$$\rho(s) = \frac{1}{Z} e^{-E(s)/kT} \quad \text{AND} \qquad [2.6]$$

$$\int_{\text{ALL PHASE SPACE}} \rho(s) = 1 .$$

(2.21) EQN (2.20) GIVES THE PROBABILITY OF SOME MICROSTATE OF THE SYSTEM. IF, INSTEAD, WE WANT TO KNOW THE VALUE OF A <u>MACROSCOPIC</u> QUANTITY LIKE THE ENERGY OF THE SYSTEM, IT WILL SIMPLY BE THE SUM OVER MICROSTATES OF THE PRODUCTS OF EACH MICROSTATE ENERGY WITH ITS PROBABILITY. EQN (2.21) IS THE GENERALIZATION OF THIS SUM TO AN INTEGRAL OVER AN ALLOWED REGION OF PHASE SPACE ($d\Gamma_s$).

(2.22) SINCE THE SYSTEM MUST BE IN <u>SOME</u> STATE (MUST HAVE SOME LOCATION IN ALLOWED PHASE SPACE), THE INTEGRAL OF THE PROBABILITY DENSITY OVER ALL OF ALLOWED PHASE SPACE MUST BE 1.

$$\int \rho(E)\, d\Gamma_s = \int \frac{1}{Z} e^{-E/kT}\, d\Gamma_s = \frac{1}{Z}\int e^{-E/kT}\, d\Gamma_s = 1 \quad [2.7]$$

OR

$$\int e^{-E/kT}\, d\Gamma_s = Z \qquad (2.22) \qquad [2.8]$$

(2.23) DEFINING THE FUNCTION $W(E) = \rho(E)\left(\frac{d\Gamma_s}{dE}\right)$, THEN FROM [2.7]

$$\int_{\text{ALL } E} W(E)\, dE = 1$$

RECALL THE MEAN VALUE THEOREM FOR INTEGRALS: [IF $f(x)$ IS CONTINUOUS ON $[a,b]$, THERE EXISTS A ξ IN (a,b) SUCH THAT $\int_a^b f(x)\,dx = (b-a) f(\xi)$.] HERE $b-a$ CORRESPONDS TO ΔE, ξ TO U AND f TO W. SO THERE EXISTS SOME ENERGY U, SUCH THAT

$$\int_{\Delta E} W(E)\, dE = (\Delta E)(W(U)) = 1 \qquad (2.23)$$

(2.24) FOLLOWS BECAUSE $\Delta\Gamma_s$ IS THE ALLOWED REGION OF PHASE SPACE. IT CORRESPONDS TO AN ENERGY RANGE ΔE.

(2.25) WE ASSUME HERE THAT ΔE IS SMALL AND THEREFORE, BECAUSE ALL MICROSTATES WITH THE SAME ENERGY ARE EQUALLY PROBABLE, ALL MICROSTATES WITHIN $\Delta\Gamma_s$ ARE APPROXIMATELY EQUALLY LIKELY.

THE REMAINDER OF §2.5.1 IS A DETAILED ANALYSIS OF AN IDEAL GAS TREATED AS A CANONICAL ENSEMBLE. THE MOST IMPORTANT RESULT IS THE EQUIPARTITION THEOREM OF CLASSICAL THERMO THAT STATES THAT, AT TEMPERATURE T THE AVERAGE THERMAL ENERGY OF ANY QUADRATIC DEGREE OF FREEDOM IS $KT/2$. THEY ALSO MENTION STIRLING'S FORMULA, THE SIMPLIFIED VERSION OF WHICH IS HANDY TO KNOW:

$$\ln(N!) \approx N \ln N - N.$$

SECTION 2.5.2 "DERIVES" THE MAXWELL DISTRIBUTION, EQN. (2.44), AND SHOWS IT HAS A PEAK AT $v_{mp} = \sqrt{2KT/m}$. OTHER HANDY VELOCITIES ARE AVERAGE VELOCITY: $\bar{v} = \sqrt{8KT/\pi m}$ AND RMS VELOCITY: $v_{RMS} = \sqrt{3KT/m}$.

§2.5.3 DESCRIBES A "GRAND CANONICAL ENSEMBLE". THE CANONICAL ENSEMBLE EXCHANGES ENERGY WITH A RESERVOIR AT A CONSTANT TEMPERATURE, A USEFUL MODEL OF THINGS RANGING FROM ATOMS TO REFRIGERATORS. THE "GRAND" VERSION CAN ALSO EXCHANGE MASS, THOUGH NONE LEAVES OR ENTERS THE COMBINED SYSTEM/RESERVOIR. THIS RESULTS IN "GIBBS FACTORS" (THAT ARE SUMMED INTO THE "GRAND PARTITION FUNCTION" OF (2.45)). THEY ARE AN EXTENSION OF THE BOLTZMAN FACTORS TO INCLUDE MASS EXCHANGE. GIBBS FACTORS ARE MOST APPLICABLE TO DENSE SYSTEMS WHERE SEVERAL PARTICLES HAVE A REASONABLE CHANCE OF TRYING TO OCCUPY THE SAME STATE. THEIR SUCCESS DEPENDS ON THEIR NATURE AS BOSONS OR FERMIONS.

§2.6 COVERS "SHANNON ENTROPY" AND "INFORMATION", THAT WERE ORIGINALLY APPLIED IN TELECOMMUNICATIONS. THE INFORMATION EXTRACTED FROM A SYSTEM (RECEIVED OVER A TELEPHONE CABLE) IS EQUAL TO THE DECREASE IN ITS SHANNON ENTROPY.

§2.7 IS SELF EXPLANATORY.

2 The Second Law

The previous chapter explored the law of energy conservation as it applies to thermodynamic systems. It also introduced the concepts of heat, work, and temperature. However, some very fundamental questions remain unanswered: What is temperature, *really*, and why does heat flow spontaneously from a hotter object to a cooler object, never the other way? More generally, why do so many thermodynamic processes happen in one direction but never the reverse? This is the Big Question of thermal physics, which we now set out to answer.

In brief, the answer is this: Irreversible processes are not *inevitable*, they are just overwhelmingly *probable*. For instance, when heat flows from a hot object to a cooler object, the energy is just moving around more or less randomly. After we wait a while, the chances are overwhelming that we will find the energy distributed more "uniformly" (in a sense that I will make precise later) among all the parts of a system. "Temperature" is a way of quantifying the tendency of energy to enter or leave an object during the course of these random rearrangements.

To make these ideas precise, we need to study *how* systems store energy, and learn to count all the ways that the energy might be arranged. The mathematics of counting ways of arranging things is called **combinatorics**, and this chapter begins with a brief introduction to this subject.

2.1 Two-State Systems

Suppose that I flip three coins: a penny, a nickel, and a dime. How many possible outcomes are there? Not very many, so I've listed them all explicitly in Table 2.1. By this brute-force method, I count *eight* possible outcomes. If the coins are fair, each outcome is equally probable, so the probability of getting three heads or three tails is one in eight. There are three different ways of getting two heads and a tail, so the probability of getting exactly two heads is 3/8, as is the probability of

Penny	Nickel	Dime
H	H	H
H	H	T
H	T	H
T	H	H
H	T	T
T	H	T
T	T	H
T	T	T

Table 2.1. A list of all possible "microstates" of a set of three coins (where H is for heads and T is for tails).

getting exactly one head and two tails.

Now let me introduce some fancy terminology. Each of the eight different outcomes is called a **microstate**. In general, to specify the microstate of a system, we must specify the state of each individual particle, in this case the state of each coin. If we specify the state more generally, by merely saying how *many* heads or tails there are, we call it a **macrostate**. Of course, if you know the microstate of the system (say HHT), then you also know its macrostate (in this case, two heads). But the reverse is not true: Knowing that there are exactly two heads does *not* tell you the state of each coin, since there are three microstates corresponding to this macrostate. The number of microstates corresponding to a given macrostate is called the **multiplicity** of that macrostate, in this case 3.

The symbol I'll use for multiplicity is the Greek letter capital omega, Ω. In the example of the three coins, $\Omega(3\text{ heads}) = 1$, $\Omega(2\text{ heads}) = 3$, $\Omega(1\text{ head}) = 3$, and $\Omega(0\text{ heads}) = 1$. Note that the total multiplicity of all four macrostates is $1 + 3 + 3 + 1 = 8$, the total number of microstates. I'll call this quantity $\Omega(\text{all})$. Then the *probability* of any particular macrostate can be written

$$\text{probability of } n \text{ heads} = \frac{\Omega(n)}{\Omega(\text{all})}. \tag{2.1}$$

For instance, the probability of getting 2 heads is $\Omega(2)/\Omega(\text{all}) = 3/8$. Again, I'm assuming here that the coins are fair, so that all 8 microstates are equally probable.

To make things a little more interesting, suppose now that there are not just three coins but 100. The total number of *micro*states is now very large: 2^{100}, since each of the 100 coins has two possible states. The number of *macro*states, however, is only 101: 0 heads, 1 head, ... up to 100 heads. What about the multiplicities of these macrostates?

Let's start with the 0-heads macrostate. If there are zero heads, then every coin faces tails-up, so the exact microstate has been specified, that is, $\Omega(0) = 1$.

What if there is exactly one head? Well, the heads-up coin could be the first one, or the second one, etc., up to the 100th one; that is, there are exactly 100 possible microstates: $\Omega(1) = 100$. If you imagine all the coins starting heads-down, then $\Omega(1)$ is the number of ways of *choosing* one of them to turn over.

To find $\Omega(2)$, consider the number of ways of choosing two coins to turn heads-up. You have 100 choices for the first coin, and for each of these choices you have

99 remaining choices for the second coin. But you could choose any pair in either order, so the number of *distinct* pairs is

$$\Omega(2) = \frac{100 \cdot 99}{2}. \tag{2.2}$$

If you're going to turn three coins heads-up, you have 100 choices for the first, 99 for the second, and 98 for the third. But any triplet could be chosen in several ways: 3 choices for which one to flip first, and for each of these, 2 choices for which to flip second. Thus, the number of distinct triplets is

$$\Omega(3) = \frac{100 \cdot 99 \cdot 98}{3 \cdot 2}. \tag{2.3}$$

Perhaps you can now see the pattern. To find $\Omega(n)$, we write the product of n factors, starting with 100 and counting down, in the numerator. Then we divide by the product of n factors, starting with n and counting down to 1:

$$\Omega(n) = \frac{100 \cdot 99 \cdots (100 - n + 1)}{n \cdots 2 \cdot 1}. \tag{2.4}$$

The denominator is just n-factorial, denoted "$n!$". We can also write the numerator in terms of factorials, as $100!/(100-n)!$. (Imagine writing the product of all integers from 100 down to 1, then canceling all but the first n of them.) Thus the general formula can be written

$$\Omega(n) = \frac{100!}{n! \cdot (100 - n)!} \equiv \binom{100}{n}. \tag{2.5}$$

The last expression is just a standard abbreviation for this quantity, sometimes spoken "100 choose n"—the number of different ways of choosing n items out of 100, or the number of "combinations" of n items chosen from 100.

If instead there are N coins, the multiplicity of the macrostate with n heads is

$$\Omega(N, n) = \frac{N!}{n! \cdot (N - n)!} = \binom{N}{n}, \tag{2.6}$$

the number of ways of choosing n objects out of N.

Problem 2.1. Suppose you flip four fair coins.

- **(a)** Make a list of all the possible outcomes, as in Table 2.1.
- **(b)** Make a list of all the different "macrostates" and their probabilities.
- **(c)** Compute the multiplicity of each macrostate using the combinatorial formula 2.6, and check that these results agree with what you got by brute-force counting.

Problem 2.2. Suppose you flip 20 fair coins.

- **(a)** How many possible outcomes (microstates) are there?
- **(b)** What is the probability of getting the sequence HTHHTTTHTHHHTHHHHTHT (in exactly that order)?
- **(c)** What is the probability of getting 12 heads and 8 tails (in any order)?