Lie Groups in Physics - P. and C. Kunasz (05/03/2012)

INTRODUCTION

Part I, which provides some quantum theory and group theory background, is based on a number of sources, including L. Schiff's **Quantum Mechanics**, 3rd ed. Part II, in which Lie groups are discussed in physical context, comes mostly from Bruce Schumm's very interesting book, **Deep Down Things.**

PART I: BACKGROUND

In 1801 Thomas Young did the famous double slit experiment, which demonstrated that light interferes with itself, like a wave. This seemed to settle an old dispute going back to the time of Newton and Huygens, c. 1680.

A. A series of experiments and observations, starting with Balmer (1885) and Rydberg (1890), and then Ritz (1908), provided evidence that the atom is quantized in energy and magnetic properties. In 1913 Bohr found the formula for the quantized "bound state" energy of the hydrogen atom, $E_n = 13.6$ eV / n^2, where n is the quantum number n=1, 2, 3, ….. This explained spectral line energies as due to bound-bound transitions:

$$E_{nm} = h\nu = 13.6 \text{ eV } (1/n^2 - 1/m^2)$$

B. In1900 Max Planck confronted the fact that thermal electromagnetic radiation has a spectrum that cannot be predicted from the assumption that light is a wave. He made the trial assumption that light is emitted in quanta and derived his empirically successful formula for the spectrum of black body emission, now known as the Planck function:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} / (e^{h\nu/kT} - 1) \quad \text{erg / Hz}$$

This formula was the debut of Planck's constant, h, the coupling constant between energy and rate of oscillation. It is more useful in the form called "h-bar":

$$\hbar = \frac{h}{2\pi} = 1.054 \text{ x } 10^{-27} \text{ erg-s} = 6.59 \text{ x } 10^{-7} \text{ eV/Hz} = 6.59 \text{ x } 10^{-7} \text{ GeV/GHz} .$$

Note that Boltzmann's constant, k, is analogous to h for temperature. When multiplied by temperature it yields energy per particle per degree of freedom. So, $h\nu/kT$ is dimensionless as it should be in the argument of the exponential. In what follows, the letter "k" will be used for wave number, not Boltzmann's constant.

C. In 1904 Einstein demonstrated the photoelectric effect, which confirmed that light can act as a collection of particles with quantum energy $h\nu$. He coined the term "photon" for a quantum of light. To get monochromatic light to activate a physical process with energy threshold E, the light must have $h\nu \geq$ E.

D. In 1924 de Broglie suggested that if light can act as photons, then perhaps a particle, like an electron, could behave like a wave, a matter wave. For light we find a relationship between wavelength, λ, and momentum, p, which should also hold for a particle,

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \text{ or, } \lambda = \frac{h}{p}; \text{ if k=}\frac{2\pi}{\lambda}, \text{and } \omega \equiv 2\pi\nu, \text{ then } p = \hbar \text{k}, E = \hbar\omega.$$

Wave number, k, and angular frequency, ω, are commonly used instead of λ and ν.

E. In 1927 Davisson, Germer, and Thomson observed electrons diffracting in crystals, confirming that electrons exhibit wave nature in this context. How big can a "particle" be and still be wavelike? Buckyballs (Carbon 60 molecules) have been shown to interfere like waves. Momentum, wavelength, energy, frequency – all these quantities are relative to the observer, not intrinsic to the particle. The only intrinsic energy-momentum property is the rest mass m, or the rest energy, $E = mc^2$.

EQUATION FOR A WAVE PACKET

A. After the discoveries above, physicists knew they had to come up with a new physical theory to replace Newton's laws and the assumption that particles and waves never share each other's characteristics. The new image of a photon or freely propagating particle is that of a wave packet, something wavy yet bunched up in space, a ***wavicle***. A wave packet must be composed of a spectrum of waves of different wavelengths (see Schumm's graphic). Classical wave mechanics tells us that the more bunched up a wave packet is in coordinate space, the broader it must be in momentum space, and vice versa. The most general mathematical form for the sinusoidal waves is the complex form:

$$\Psi(x,t) = \exp(i(\text{k}x - \omega t))$$

This is justified by our conclusion below that the wave equation that works for quantum physics will not admit a real solution. Note that this traveling wave has constant phase when $x = x_0 + (p/m)\,t$, as expected.

B. At this point no one knew what the "substance" of such waves would be, or how they would connect to physical observables, but they knew that a method for finding them would have to be some kind of "wave equation". Various wave equations were very well known from classical mechanics. Which one(s) would work for the linear superposition of waves of differing wavelengths in a wave packet? It became clear that: 1) the new equation must be linear, 2) it must not have wavelength or energy in its coefficients because you want one equation for all the superposed waves in the wave packet. If you also assume a relatively simple equation, and write it in one dimension x, you are restricted to the form: $\frac{\partial \Psi}{\partial t} = \gamma \frac{\partial^2 \Psi}{\partial x^2}$.

It was this approach that led Schrodinger (1925) to his well known wave equation, referred to here as (SE), for the free particle:

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \qquad \text{(SE)}$$

We can differentiate our simple wave function and see how momentum and energy can be associated with their differential operators: $i\hbar\frac{\partial\Psi}{\partial t}=E\Psi$ and $-i\hbar\frac{\partial\Psi}{\partial x}=p\Psi$

Thus we have the association: $i\hbar\frac{\partial}{\partial t}\leftrightarrow E, \quad -i\hbar\frac{\partial}{\partial x}\leftrightarrow p$

This association is the cornerstone for most of the theorems and results of quantum theory. Schrodinger showed his equation to be equivalent to and handier than Heisenberg's matrix formulation of quantum mechanics, and Feynman later showed it to be equivalent to his Feynman diagrams, which provided a means to calculate probabilities for processes like the decay of a high energy particle in which the fluctuating vacuum with its virtual particles play a central role.

The Schrodinger equation (SE) above is a statement of energy conservation for the free particle, free meaning not being acted upon by a field of potential energy. It says total energy equals kinetic energy:

$$E\Psi=(p^2/2m)\Psi.$$

But, in almost all realistic cases a particle is not free, like an electron in orbit, or a quark in a proton. As it propagates in a potential energy field V(x,t), it experiences the force -d/dx V(x,t). So, physicists took a leap of faith and added a general potential energy to the right-hand-side of SE,

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+V(x,t)\Psi(x,t)$$

The solution to the SE now depends on which potential energy V(x,t) is under consideration. Most textbook examples are not easy, requiring special functions, infinite series, and so on. Most problems beyond the three or four easiest require elaborate math and some set of well chosen approximations, the art of approximation being the craft of the theoretician.

INTERPRETATION OF THE WAVE FUNCTION

The solution, $\psi(x,t)$, to the SE is in general a complex function (one factor of which is a complex unit vector oscillating with angular frequency ω for total energy of $\hbar\omega$. One can pose the question, "What physical interpretation can be given this complex function?" The fact that it is complex means that it cannot be given a direct identification with any measurable quantity. In 1924, Max Born suggested:

1) $P(x,t) = \Psi^*(x,t)\Psi(x,t) =$ the probability per unit volume for the particle to be found at spacetime point (x,t). The complex conjugate (star) assures us that P is a positive real number. Thus, some refer to $\Psi(x,t)$ as "the square root of reality".

2) $<Q> = \iiint \Psi^* Q \Psi d^3r =$ expectation value for quantity Q. In this context the quantity Q is thought of as an "operator". Some operators are multiplicative, like position x. Others are differential operators, like energy and momentum operators, described above.

The expectation value for momentum on the x axis is: $<p> = \iiint \Psi^* (-i\hbar \partial/\partial x)\Psi d^3r$

UNCERTAINTY PRINCIPLE AND GOD THROWING DICE

A. Uncertainty. Heisenberg pointed out that in wave mechanics $\Delta x \Delta k \geq 1$, where the deltas refer to the spread of the wave packet in space and wavelength.

Using our quantum relationships above we have: $\Delta x \Delta p \geq \hbar$

This is Heisenberg's Uncertainty Principle. Using a rigorous definition of Δx and Δp we find $\Delta x \Delta p \geq \hbar/2$. Similarly, $\Delta E \Delta t \geq \hbar/2$, where E is energy and t is time.

These relationships were first interpreted as statements about how when you measure one thing you interfere with another. Later it was understood to be a statement about knowledge itself.

B. Example 1: solution of SE for the central force of attraction known as Coulomb's Law. This is a simplified hydrogen atom. Mathematically we find multiple solutions, called "states" of the atom. Any linear combination of these solutions is also a solution. This implies that an atom can be prepared in one state or be in a probabilistic distribution over all of them.

For the hydrogen atom with its unchanging central force field, the solution separates as the product of three functions: one over time, one over angle, and one over radial distance from the nucleus. The first three discrete spatial wave functions for the H-atom, are common illustrations in high school chemistry texts.

C. Example 2: Probabilistic transmission coefficient for a free particle hitting an energy barrier that a classical particle cannot surmount. This is called "tunneling". Tunneling lies at the heart of many sub-atomic phenomena. It allows nuclear transitions like alpha radioactivity to occur, it threatens to limit the quest for nanoscale computer circuitry, and it enables a number of high tech applications, like the scanning tunneling microscope that "sees" individual atoms. Though the quantum probability of passing unscathed through a macroscopic barrier is exceedingly low, in old Tibet the name Gomang Monastery meant "monastery of many doors" because, they say, its well-trained meditators could walk straight through the walls.

SYMMETRIES OF THE SHRODINGER EQUATION (SE)

A. Noether's Theorem: every symmetry or invariance in physics is associated with the conservation of some property. This is easy to see in F=MA, but it is not obvious in other theories. The proof can be complicated.

B. Assume potential $V(r,t) = V(r)$, as in the H-atom, with r being radial distance from the nucleus. Then the solution separates into a product of a radial function, $u(r)$, an angular function, $Y_{lm}(\vec{\Omega})$, and a complex phase over the oscillation cycle

$$\Psi(\vec{r},t) = u(r)Y_{lm}(\vec{\Omega})\exp(i\omega t)$$

A complex phase also occurs in the azimuthal angular function. But, wherever it occurs in the wave function, we can think of the wave function value at any point of space-time as a vector in the complex plane. It is not hard to see that only the modulus (length) of this vector is important to the physical results. No change will occur in any physical quantity if we apply an arbitrary phase shift to this complex vector so long as we do it equally for all points in space-time.

C. There are mathematical proofs for the following symmetries:

* The SE is **invariant under spatial displacements if and only if momentum is conserved**. The free particle is an example. An electron in a hydrogen atom is not.

* The SE is **invariant under displacements in time if and only if energy is conserved**.

* The SE is **invariant under changes in spatial orientation about a given axis if and only if angular momentum about the axis is conserved**.

* The SE is **symmetric under mirror reflections if and only if parity is conserved**. Parity being conserved means that under spatial reflection the wave function does not change sign.

GROUPS

A group is a set of elements and a rule for binary combination of these elements that obeys four axioms. Example: the set of spatial displacement vectors $\vec{\rho}$ is a set of vectors that form a group under the binary operation of vector addition. Here are the defining properties of a group G as illustrated in this case:

1. $\vec{\rho}_1, \vec{\rho}_2 \in G \rightarrow \vec{\rho}_1 + \vec{\rho}_2 \in G$	Closure
2. $\vec{0} + \vec{\rho}_1 = \vec{\rho}_1 + \vec{0} = \vec{\rho}_1$ for all $\vec{\rho}_1 \in G$	Identity
3. For every $\vec{\rho}_1 \in G$, there exists $\vec{\rho}_2 \in G$ such that $\vec{\rho}_1 + \vec{\rho}_2 = \vec{0}$	Inverse
4. $(\vec{\rho}_1 + \vec{\rho}_2) + \vec{\rho}_3 = \vec{\rho}_1 + (\vec{\rho}_2 + \vec{\rho}_3)$ for all $\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3 \in G$	Associativity

These groups of vectors can be parametrized by continuous variables, the numerical coordinates of the spatial displacement vectors. They are smooth functions of these variables.

Ways to categorize groups:

* A group is *finite* or *infinite*.

Examples of finite groups: The integers modulo n under addition
The six element group of rotations by 60 degrees under addition
The mirror reflection group: {I,R}, IR=RI=R, RR=I
Examples of infinite groups: The integers under addition, but not multiplication
The reals under addition, or multiplication if zero is excluded (it has no inverse).
R(2), R(3) (see below)
The set of unit complex vectors under multiplication

* A group is called *abelian* if ab = ba for all elements a, b in the group. True for rotations in the plane, R(2), but not in three dimensions, R(3). If ab = ba then a and b are said to *commute*. A *commutative group* is one in which all products commute. The *commutator bracket* [a,b] = (ab-ba) lies at the heart of quantum theory and particle physics. The commutator bracket is usually discussed in the matrix formulation of quantum mechanics where it relates to products of matrices theorems of matrix algebra.

R(2) and R(3) are the groups of rotations (orientations) in 2-D and 3-D respectively. Rotations can be thought of as vectors for which length is rotation angle and direction is axis of clockwise rotation. The operation for the rotations in the groups R(2) and R(3) is that of addition, which means successive rotations. The operation, sometimes called "product" (r1 r2) is "rotation 2 followed by rotation 1". In R(2), r1 r2 = r2 r1. In R(3) this is not true in general. R(3), like SU(3) is not commutative. In R(3) these vectors comprise a sphere of radius pi with all diametrically opposing points on the boundary being equivalent. U(1) is the 1-D unitary group. It can be modeled as the set of numbers exp(iα) under multiplication, i.e. the complex numbers of unit modulus.

* A group is *continuous* if its elements can be labeled by one or more continuously varying parameters. For example, a rotation in R(3) can be thought of as a vector whose projections on three orthonormal unit vectors are the smoothly varying parameters.

* **Definition of Lie group L**: For $a,b \in L$, the parameters of product c = ab must be smooth, differentiable functions of the parameters of a and b. Most of the groups of interest in physics are Lie groups: space displacement, time displacement, complex phase angle, spatial rotation, and (we will see) rotation in an abstract *isospin space.* The group of unit modulus complex numbers, exp($\iota\alpha$), known as U(1) is a smooth function of parameter angle α. A physical group which is not a Lie group is the group of mirror reflections.

* A group is *continuously connected* if any two group elements can be connected by continuous variation of the group parameters.

* Imagine the set of displacements in the (x,y) plane. Any two elements of the group of displacement vectors can be connected by an infinite number of paths in the plane which can be smoothly morphed into one another. A group is <u>*simply connected*</u> if there is only one such set of paths through parameter space. If there are, for example, two such distinct sets, the group is <u>*doubly connected*</u>.

* The <u>*generators*</u> of a group are a minimal set of group elements from which all elements of the group can be built up by the group operation. The number of generators is called the <u>*dimension*</u> of the group; e.g. R(2) has one generator, while R(3) has three generators, as rotation vectors in 3-D can be generated from the three unit basis elements on the Cartesian axes.

* A group is *compact* if every infinite sequence of elements has a limit element that is also in the group.

It can be seen that the set of spatial displacement vectors under the operation of vector addition form an abelian, continuously connected, three-parameter, compact Lie group. The Schrodinger equation for a free particle is invariant under the Lie group of spatial displacements. That means that if the spatially displaced wave function $\psi(x- x_0,t)$ is a valid solution over the time variable, then it describes a particle with constant linear momentum. Theorems like this are formally proved in most graduate level texts. The set of Lorentz transformations form a continuous, six-parameter Lie group that is not compact. Why not? Is it abelian? The 3-D rotation group R(3) is continuously connected, and doubly connected.

PART II: ABSTRACT PHYSICAL GROUPS IN PARTICLE PHYSICS

Groups like displacements in time, space, and angle are not abstract. They refer to real, physical, or geometric entities that are intuitively understandable. But, physicists, starting with Heisenberg, have discovered abstract symmetries that take the form of rotation groups under which a force of nature, and thus the physical model is invariant. The prototype example of this is the complex wave function itself. The physical model is invariant over arbitrary global shifts in its phase angle. This space of phase freedom can be represented as U(1), the unitary Lie group of one parameter. Even though this is a group of complex numbers, which then might be thought to have two free parameters, the real and imaginary parts of the complex unit vector, it has the constraint of unitarity (the complex numbers must have unit modulus), so this leaves one free parameter.

Heisenberg's historical example is the invariance of the strong force between two nucleons (protons and neutrons) with respect to which nucleon types are interacting: NN, PP, or NP. The set {N,P} is called the neutron proton doublet. Picture a circle drawn on an abstract plane, called the *doublet isospin space*, with the neutron at one pole and the proton at the other. The Lie group of rotations around this circle can be represented as a complex unit vector free to rotate in the complex plane. So, we now have two independent complex spaces: that of the phase angle of the wave function, and that of isospin. When these two complex rotations are combined in the total wave function, the result is special unitary group SU(2), a

Lie group that has three generators. In the matrix analog, SU(2) is the subgroup of U(2) with determinant = 1, where U(2) is the set of real 2x2 matrices. There will be more on this below.

The weak force has similar behavior. The generational doublets are the pairs of particles experiencing the weak force which easily transform into each other: the electron neutrino and electron, the mu neutrino and muon, the tau neutrino and tauon, and the quark pairs: (u,d), (c,s), (t,b). **All of these particles carry the same weak force charge, or coupling strength**. The parameters for the Lie group for this model of the weak force include the complex phase angle in the wave function plus the independent complex phase of the rotation in *weak isospin spac*e. Again, the two independent complex rotations comprise the Lie group SU(2).

YANG-MILLS GAUGE THEORY

The idea introduced by Yang and Mills in 1954, often called **gauge theory**, is a way of translating the primitive symmetries of nature into potentials, and forces, or at least energetic identity transformations. It is thus a dynamical theory starting with a symmetry expressed as a physical invariance under the actions of a Lie group, and concluding with forces or energetic interactions. It must be differently implemented for each of the forces of nature in the Standard Model (EM, weak, and strong). After it was introduced it was thought to have failed because it required its force carriers to be massless. Yang-Mills gauge theory was later revived when symmetry breaking was introduced, and it now serves as the intellectual cornerstone of the Standard Model. It has its simplest Lie group in the electromagnetic force: U(1). A more complicated one, SU(2), is found for the weak force, and the most complicated one, SU(3), is found for the strong force.

The gauge idea is based on the observation that though all measurable physical quantities are independent of arbitrary *global* complex phase shifts in the purely mathematical sense, phase changes occurring globally over space-like events are not acceptable in special relativity. Space-like events are those for which the spatial separation is greater than the distance light would travel over the time separation. For example, if I change my phase angle here and now, my buddies vacationing on Mars are going to need some time to hear about it and respond by matching my action there, but by that time I may have chosen another phase angle, and we will always be out of whack. This worried Yang and Mills, so they set out to investigate *local* phase variations, ones that vary over space and time. But, the spatial derivative in the SE picks up this new spatial gradient, and the SE no longer gives the correct solution. So, the effect of the extra spatial variation must be offset by the addition of one or more compensating "potentials" in the SE. It is these compensating potentials, which vary from point to point as the local phases vary, which provides the dynamics or energetics of the interaction in question.

In electromagnetism the gauge freedom is the phase freedom in the complex wave function, as represented by the Lie group U(1). It turns out that the needed potential is exactly that of classical EM theory. It is the four-vector potential, $(\vec{A},\phi)$, that yields the observables E, and B (electric and magnetic fields) over a wide range of gauge freedom. In one useful gauge relationship between A and ϕ called the Coulomb gauge, we have $\phi = 0$, and the electric and magnetic fields (physical observables) are

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \;;\; \vec{B} = \vec{\nabla} \times \vec{A}$$

One has a great deal of freedom in the definition of vector A without changing the electric or magnetic field.

In the case of the weak force, the symmetry is more complicated. It has a part just like the complex angle discussed above, but it also has invariance over an abstract *weak isospin space* defined by rotation in the generational doublets. As above, they come in pairs: (electron neutrino, electron), (muon neutrino, muon), (tau neutrino, taon), (up quark, down quark), (charmed quark, strange quark), and (top quark, bottom quark). The weak force charge is identical over all these particles, and one complex unit vector is used to represent all pairs. The Lie group arising from the two independent complex angles is SU(2), which has three generators. It can be shown mathematically that for each Lie group generator there must be a potential to compensate. The weak force therefore requires for local phase compensation at least the sum of three independent potential functions multiplied into the weak charge g and the wave function: $(W_1(x)+W_2(x)+W_3(x))\; g\; \Psi(x,t)$

But, things get yet more complicated. Additional compensating functions must be added because the weak force Lie group SU(2) is not commutative. The so called “Lie algebra” of commutations relations among the generators of the Lie group gives rise to yet more potential terms in the equation, one for each non-zero commutation bracket. These extra functions represent interactions among the force carriers themselves: W^+, W^-, and Z^0. In Feynmann diagrams each compensating function gives rise to a *minimum interaction vertex*. See Schumm’s graphics.

Another way of looking at compensating functions and elementary processes is to realize that each compensating function for each generator of each Lie group gives rise to its own *minimum interaction vertex* (see Schumm’s graphics). A compensating function that arises from a non-zero commutation bracket gives rise to a minimum interaction vertex involving force carriers only. From these various minimum interaction vertices Feynmann diagrams of arbitrarily great complexity can be constructed in pursuit of an accurate result.

How do these force carrier bosons arise from the theory? In other words, how do we go from fields to quanta? Quanta come about in the extension of quantum mechanics called quantum field theory (QFT). This is a theory based on a relativistic form of the SE that also includes creation and annihilation operators in addition to the normal operators for position, momentum, angular momentum, and energy. In this somewhat ad hoc way fields generate particles in full consistency with conservation of mass-energy. In principle, from QFT one could calculate separate probabilities for each of the myriad ways the energy of two colliding quarks in the LHC could manifest into specific sets of hadrons, leptons, and bosons. Such computations are notoriously difficult. As Bruce Schumm writes in “**Deep Down Things”**, he would rather this be done by somebody who actually gets paid to do it.

In what lies above we have not discussed the fact that the theory of the weak force must be amended by the concept of symmetry breaking. That lies outside the scope of this essay. We also have not discussed, with the exception of the early Heisenberg model, Lie groups in the context of the strong force. For the strong force, there is no compromising with Lie group theory, as there is in the weak force. There is no symmetry breaking, or massive bosons like the Higgs waiting to be discovered in the lab.

So, how do Lie groups work in the strong interaction? The concept is the same as for the weak force: a physical property is found that remains constant as rotation occurs in abstract spaces. The physical property is strong force charge, called the *color charge*. Again, you could also say it is the physical model itself. Now, where are the abstract complex spaces? As found by Fritsch and Gell-Mann, each of the three "colors" of the strong force color charge gives rise to an abstract complex plane in which a rotation can occur without any effect on the physical model. These three independent complex rotation spaces form the axes of a complex 3-D space in which rotations can occur. This space is the Lie group SU(3), the rotations in that space. The special unitary group, SU(3), has eight (3x3-1) generators and a complicated *Lie algebra* of commutation relations. These commutation relations must be compensated by extra functions in the SE, or its relativistic counterpart, and each such function gives rise to a new minimum interaction vertex. Each generator of SU(3) induces one color-charged force carrier particle through the action of quantum field theory. These massless bosons are the eight gluons described in much of the popular literature on particle physics. They comprise 98% of the mass energy of ordinary matter.

The discovery by Fritsch and Gell-Mann of the SU(3) group in the strong interaction came from their study of the patterns formed by newly discovered high energy particles in a plot of isospin vs "hypercharge", or isospin vs strangeness. It resulted in the successful prediction of the omega minus particle, composed of three strange quarks.

In Bruce Schumm's book, Deep Down Things, gauge theory and the mathematical structures it employs are presented as the theoretical foundation in the Standard Model of modern particle physics. Through the application of gauge theory to the electroweak and strong forces the Standard Model correctly explains known interactions, and has predicted new high energy particles. It also allows renormalization, something no other known theory does.

In gauge theory the underlying mathematical structures are the Lie groups U(1), SU(2), SU(3), and objects of higher rank. The weak force and its Lie group SU(2) are the primary focus of Schumm's book, but his description of the group SU(2) leaves something to be desired. We learn that it is somehow a rotation involving complex numbers on a two-dimensional complex plane. The action of a group element performs a complex rotation on the complex wave function in such a way that if the rotation is done globally over space and time then no measurement can detect it. If the complex rotation is allowed to vary from point to point, the effects of this variation must be countered by new potentials in the wave equation. These potentials describe new fields and their quanta (particles), and the interactions they undergo or mediate, such as the decay of the pion or neutron, or the scattering of a neutrino.

Schumm's book attempts to describe the rotational action of SU(2) using little more than the Pythagorean relation between two complex moduli, leaving the reader with a desire to get to the bottom of it. So, the purpose of these notes is to present some supplementary material which will I. connect the formal definition of SU(2) to Schumm's description, and II. connect SU(2) to the rotation operator in quantum mechanics. In addition, in section III. quaternions are presented as a group of 2x2 complex matrices related to SU(2) which employ a half-angle argument, are used to represent complex rotations, and ~~to~~ combine as an operator to rotate a vector in real 3-space.

I. SU(2).

We will show that Schumm's verbal description of SU(2) conforms to the standard mathematical definition.

Definition of SU(2): the group of unitary 2x2 matrices with determinant (det) = 1. This is the group

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & -\bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\},$$

where $\bar{\alpha}$ is the complex conjugate of α, etc.

and ~~Where~~ C is the set of complex numbers. We can rewrite this definition using the polar form of the complex numbers

$$\left\{ \begin{pmatrix} a e^{i\theta} & -b e^{-i\phi} \\ b e^{i\phi} & -a e^{-i\theta} \end{pmatrix} : a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\},$$

where $\mathbb{R}$ = the set of real numbers, and $a, b \geq 0$.

Now we identify a and b as Schumm's S_x and S_y, and we identify θ and ϕ as his two complex phases.

We use notation $\begin{pmatrix}1\\0\end{pmatrix}$ for the pure spin up state, and $\begin{pmatrix}0\\1\end{pmatrix}$ for the pure spin down state. The general wave function $\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}$ being a complex superposition of the two. Assume $|\psi_1|^2 + |\psi_2|^2 = 1$, to conserve probability.

When an element of SU(2) acts on the column vector we have, using the non-polar forms,

$$\begin{pmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\alpha\psi_1 - \bar{\beta}\psi_2\\ \beta\psi_1 + \bar{\alpha}\psi_2\end{pmatrix} \equiv \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}, \text{ and}$$

$$\begin{aligned}\psi_3\bar{\psi}_3 + \psi_4\bar{\psi}_4 &= (\alpha\psi_1 - \bar{\beta}\psi_2)(\bar{\alpha}\bar{\psi}_1 - \beta\bar{\psi}_2) + (\beta\psi_1 + \bar{\alpha}\psi_2)(\bar{\beta}\bar{\psi}_1 + \alpha\bar{\psi}_2)\\ &= \psi_1\bar{\psi}_1(\alpha\bar{\alpha} + \beta\bar{\beta}) + \psi_2\bar{\psi}_2(\alpha\bar{\alpha} + \beta\bar{\beta}) + 0 + 0\\ &= \psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 = |\psi_1|^2 + |\psi_2|^2 = 1.\end{aligned}$$

Thus, SU(2) is a Lie group of transformations which can vary complex phase angles while preserving the norm, which is the same as preserving the probability that the system is in one state or the other when measured. In Schumm's notation $S_x^2 + S_y^2 = 1$.

So, we now have the connection between Schumm's description of SU(2) and the formal definition. I have not dealt with generators and Lie algebras here. To study that see Wikipedia and other sources.

II. The Rotation Group

As a warm up for spin and angular momentum we present a proof of Noether's Theorem for conservation of linear momentum, and as a side benefit on the road to SU(2) we present a demonstration of why the wave function of the electron, and the other fermions, takes on a minus sign after a 2 pi rotation. Space itself seems to be isotropic, but the relationship of the complex wave function to it is not.

In quantum theory, an "operator" is a mathematical object which acts on the wave function to 1) produce a new wave function, such as the original one advanced 2 seconds in time, or one centimeter in space, or 35 degrees in orientation about some axis, or 2) to multiply the original wave function by a measurable physical quantity, such as energy, or momentum, or angular momentum. The Hamiltonian is the differential operator which acts on the wave function to return the total energy (kinetic plus potential) of the system. Its definition employs the identification of the momentum operator and the spatial gradient. We write these equations in one-component form, and in full vector form:

$$H = \frac{\vec{P}^2}{2m} + V = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V = i\hbar\frac{d}{dt} \qquad (0)$$

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$$P_x \to -i\hbar\frac{\partial}{\partial x} \qquad (1)$$

$$\vec{P} \to -i\hbar\vec{\nabla}.$$

For a time independent Hamiltonian the wave equation separates into time and space factors, and the wave function solution becomes the product

$$\psi(\vec{r},t) = \psi(\vec{r})\, \exp(-iEt/\hbar).$$

Using Dirac's bra-ket notation we can write the Schrodinger wave equation for a time independent energy operator, H

$$i\hbar\frac{d}{dt}\,|\psi(\vec{r})\rangle = H\,|\psi(\vec{r})\rangle \qquad (2)$$

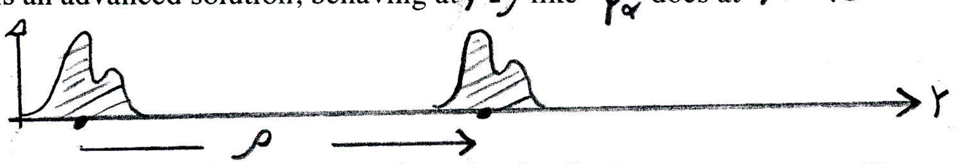
Now we consider the operator $U(\vec{\rho})$ which displaces the wave function in space as

$$U(\vec{\rho})\,\psi_\alpha(\vec{r}) \equiv \psi_{\alpha'}(\vec{r}) \equiv \psi_\alpha(\vec{r}-\vec{\rho}).$$

Then we ask whether the space displaced wave function is a solution of (2). In other words, we ask whether quantum physics is symmetric under space displacements, and what that implies.

We point out that $\psi_{\alpha'}(\vec{r}) \equiv \psi_\alpha(\vec{r}-\vec{\rho})$

is an advanced solution, behaving at $\vec{r}=\vec{\rho}$ like ψ_α does at $\vec{r}=\vec{0}$



(fig.1)

We will now derive a compact form for the displacement operator. For ease of notation we temporarily let the variation occur only on the x-axis. We will expand the displaced wave function in a Taylor series, but first recall the Taylor series for exp(-x)

$$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 \cdots$$

If we Taylor expand $\psi_\alpha(x-\rho)$ about $\rho=0$ then we can identify the expansion of $e^{-\rho\frac{\partial}{\partial x}}$ in the result

$$\psi_{\alpha'}(x) = \psi_\alpha(x-\rho) = \psi_\alpha(x) - \rho\frac{\partial}{\partial x}\psi_\alpha(x) + \frac{1}{2}\rho^2\frac{\partial^2}{\partial x^2}\psi_\alpha(x) - \cdots$$

$$= \exp\left(-\rho\frac{\partial}{\partial x}\right)\psi_\alpha(x). \qquad (3)$$

Returning to the 3-D representation and using the momentum operator, we have

$$\psi_{\alpha'}(\vec r) = \psi_\alpha(\vec r - \vec\rho) = \exp\left(\frac{-i}{\hbar}\vec\rho\cdot\vec p\right)\psi_\alpha(\vec r) \qquad (4)$$

So, the displacement operator is the complex exponential

$$U(\vec\rho) = \exp\left(\frac{-i}{\hbar}\vec\rho\cdot\vec p\right),$$

which is an infinite polynomial series of increasing powers in the product $\vec\rho\cdot\vec p$.

We now ask whether (4) is a valid solution of the Schrodinger equation (2), but first we must establish the fact that in general $\hat O\psi$ is a valid solution iff operator $\hat O$ is time independent, and $\hat O$ commutes with the Hamiltonian operator. This fact is easy to see when we plug the trial solution $\hat O\psi$ into (2).

$$i\hbar\frac{d}{dt}|\hat O\psi\rangle = \hat O\, i\hbar\frac{d}{dt}|\psi\rangle = \hat O H|\psi\rangle = H|\hat O\psi\rangle = H\hat O|\psi\rangle$$

$$-\text{OR}- \quad [\hat O H, H\hat O] = 0$$

an arbitrary

In the case of the displacement operator, commutation with the Hamiltonian is equivalent to commutation between the momentum operator $\vec p$ and the Hamiltonian because $U(\vec\rho)$ expands as a polynomial in $\vec\rho\cdot\vec p$ and $\vec\rho$ is ~~a~~ real vector. If the momentum operator commutes with the Hamiltonian the momentum is a constant of the motion. Thus we conclude that if the space displaced wave function is a solution of the wave equation then linear momentum is conserved. This stateent and its inverse constitute Noether's Theorem for linear momentum. We do not prove the inverse here.

There is a similar proof of Noether's Theorem for conservation of energy: a time displaced solution is valid iff the energy of the system is conserved.

Noether's Theorem for conservation of angular momentum as a consequence of invariance of the wave equation to rotations is more complicated, especially when spin (the intrinsic angular momentum of a particle) is included. Spin implies internal degrees of freedom in the wave function, which must then be represented as a vector or object of higher rank. Application of the rotation operator to such a wave function gets somewhat more mathematical than in the scalar case, but here goes....

We assume that the wave function is a vector $\psi_\alpha(\vec r) \longrightarrow \vec\Psi_\alpha(\vec r)$

And act on it with a rotation operator which locks the argument and wave vectors together, remembering that as in (Fig1) the angle-retarded argument $R^{-1}\vec r$ acts to advance the solution (as a function of angle).

$$U_R(\vec\phi)\vec\Psi_\alpha(r) \equiv R\vec\Psi_\alpha(R^{-1}\vec r), \qquad (5)$$

where R is the operator that rotates the operand through the rotation $\vec\phi$.

We will find the form of the rotation operator for infinitesimal angles, then integrate the resulting differential equation for an operator valid over all angles. So, assume an infinitesimal rotation ~~angle~~ $\vec{\phi}$ and write

$$U_R(\vec{\phi})\,\vec{\psi}_\alpha(\vec{r}) \cong \vec{\psi}_\alpha(\vec{r}-\vec{\phi}\times\vec{r}) + \vec{\phi}\times\vec{\psi}_\alpha(R^{-1}\vec{r}) \qquad |\vec{\phi}| \ll 1 \tag{6}$$

The first term can be Taylor expanded, then converted to momentum operator language

$$\left.\begin{aligned}\vec{\psi}_\alpha(\vec{r}-\vec{\phi}\times\vec{r}) &\cong \vec{\psi}_\alpha(\vec{r}) - \vec{\phi}\times\vec{r}\cdot\nabla\vec{\psi}_\alpha(\vec{r})\\ &= \vec{\psi}_\alpha(\vec{r}) - \frac{i}{\hbar}\vec{\phi}\times\vec{r}\cdot\vec{p}\,\vec{\psi}_\alpha(\vec{r})\end{aligned}\right\}\ \text{to first order in } \phi.$$

Using the triple scalar product identity: $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A} = \vec{C}\cdot\vec{A}\times\vec{B}$
we find

$$\vec{\psi}_\alpha(\vec{r}-\vec{\phi}\times\vec{r}) \cong \vec{\psi}_\alpha(\vec{r}) - \frac{i}{\hbar}\vec{\phi}\cdot(\vec{r}\times\vec{p})\,\vec{\psi}_\alpha(\vec{r})$$

Now we identify $\vec{r}\times\vec{p}$ as the angular momentum $\vec{L}$, and we have

$$\vec{\psi}_\alpha(\vec{r}-\vec{\phi}\times\vec{r}) \cong \vec{\psi}_\alpha(\vec{r}) - \frac{i}{\hbar}\vec{\phi}\cdot\vec{L}\;\vec{\psi}_\alpha(\vec{r}). \tag{7}$$

The second term in (6) is a vector cross-product which can be written in terms of a vector **S** whose coefficients are spin matrices.

$$\vec{\phi}\times\vec{\psi}_\alpha(R^{-1}\vec{r}) \cong \vec{\phi}\times\vec{\psi}_\alpha(\vec{r}) = \frac{-i}{\hbar}(\vec{\phi}\cdot\vec{S})\,\vec{\psi}_\alpha(\vec{r}),\ \text{where} \tag{8}$$

$$\vec{\phi}\cdot\vec{S} = \phi_x S_x + \phi_y S_y + \phi_z S_z,\ \text{and}$$

$$S_x = i\hbar\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad S_y = i\hbar\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\quad S_z = i\hbar\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}.$$

This looks confusing at first, but note that the spin matrix elements simply combine the proper components of $\vec{\phi}$ and $\vec{\psi}_\alpha$ to satisfy the definition of the vector cross-product:

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\,\hat{i} + (A_zB_x - B_zA_x)\,\hat{j} + (A_xB_y - B_xA_y)\,\hat{k}$$

$$\text{e.g. } \left(\vec{\phi}\times\vec{\psi}_\alpha(\vec{r})\right)_z = \phi_x(\psi_\alpha(r))_y - \phi_y(\psi_\alpha(r))_x,\ \text{etc.}$$

We identify $\vec{L}$ as orbital angular momentum, and $\vec{S}$ as spin. Total angular momentum is $\vec{J} = \vec{L} + \vec{S}$, and we therefore have

$$U_R(\vec{\phi}) = 1 - \frac{i}{\hbar}\vec{\phi}\cdot(\vec{L}+\vec{S}) = 1 - \frac{i}{\hbar}\vec{\phi}\cdot\vec{J} \qquad (9)$$

as the matrix operator for vanishingly small rotations, $\phi \ll 1$.

As a matrix operator $\vec{L}$ is diagonal, acting only on the spatial part of the wave function. $\vec{S}$, with its off diagonal terms, mixes up the vector components of the wave function. The components of $\vec{L}$ satisfy the well known quantum commutators for angular momentum, and $\vec{S}$ and $\vec{J}$ act the same way:

$$[L_x, L_y] = i\hbar L_z \quad \text{and cyclic permutations}$$

$$[S_x, S_y] = i\hbar S_z \quad \text{"} \quad \text{"} \quad \text{"}$$

$$[J_x, J_y] = i\hbar J_z \quad \text{"} \quad \text{"} \quad \text{"}$$

In ordinary vector spaces the cross-product equation $\vec{A} \times \vec{A} = \vec{A}$ is absurd because we remember that $\vec{A} \times \vec{B}$ must be perpendicular to $\vec{A}$ and to $\vec{B}$. But now that L, S, and J are 3-vectors whose coefficients are matrices that do not commute the equations

$$\vec{L} \times \vec{L} = i\hbar\vec{L} \qquad \vec{S}\times\vec{S} = i\hbar\vec{S} \quad \text{and} \quad \vec{J}\times\vec{J} = i\hbar\vec{J}, \qquad (10)$$

when analyzed component by component, become the commutation relations of the rotation group, as discussed by Schumm in less technical language.

Now lets integrate (9). Assume for simplicity that ϕ and J are parallel along the z-axis, and write

$$U_R(\vec{\phi}) = \left(1 - \frac{i}{\hbar}\phi J_z\right)$$

If we now assume a trial solution $U_R(\phi)$ as the operator for finite angles ϕ, and we notate the infinitesimal angle by $\Delta\phi$, we have

$$U_R(\phi + \Delta\phi) \cong \left(1 - \frac{i}{\hbar}\Delta\phi J_z\right) U_R(\phi)$$

acts first to produce finite rotation

acts second to increment ϕ by $\Delta\phi$

This differential equation is easy to solve. Subtract $U_R(\phi)$ then divide by $\Delta\phi$ to get

$$\frac{U_R(\phi+\Delta\phi) - U_R(\phi)}{\Delta\phi} \rightarrow \frac{dU_R(\phi)}{d\phi} = \frac{-i}{\hbar} J_z \qquad (11)$$

Then the solution for general rotations is

$$U_R(\phi) = \exp(-i\phi J_z/\hbar)\,. \tag{12}$$

Now we can see the effect of rotating an electron through $\phi = 2\pi$

For bosons $J_z = \hbar, 2\hbar, \ldots$ and $U_R(2\pi) = +1$

For fermions $J_z = \frac{\hbar}{2}, \frac{3\hbar}{2}, \ldots$ and $U_R(2\pi) = -1$

Turn the electron twice and you get $U(4\pi) = -1 \times -1 = +1$.

The proof of Noether's Theorem for conservation of angular momentum again comes from demanding that the rotated solution satisfies (2).

From the definitions of S_x, S_y, S_z above we can easily construct S^2 and find that it is $2\hbar^2$. A known result for angular momentum is that the total angular momentum is $\sqrt{l(l+1)}\,\hbar$, where the quantum number $l = 0, 1, 2, \ldots$. Therefore our result is good for l=1 only. This is a consequence of the fact that we assumed the wave function to be a vector.

We now concentrate on a two-state spin system with no orbital angular momentum. But first, a review of some matrix mechanics. A Hermitian matrix H is a square matrix which is equal to its Hermitian adjoint, $H^\dagger$ i.e. to its complex conjugate transpose $H = H^\dagger$. A unitary matrix is defined as having inverse equal to its Hermitian adjoint, i.e.

$$HH^\dagger = H^\dagger H = I$$

The trace Tr(M) of a matrix M is the sum of its diagonal elements. We have the useful theorem which identifies a zero trace with a determinant = +1.

$$\det(e^A) = \exp(\mathrm{Tr}(A)).$$

The group of rotation operators $U_R(\vec{\phi})$ has properties derived from the properties of the angular momentum operator $\vec{J}$. For the case of spin 1/2, and no orbital angular momentum, 2x2 complex matrices can be found for J. Without proof or derivation we present these as the matrices:

$$\vec{J} = \frac{1}{2}\hbar\vec{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(Pauli spin matrices)

These matrices are Hermitian and traceless, thus $U_R(\vec{\phi})$ is the group of 2x2 unitary matrices with det=1. This is the definition of SU(2).

III. HAMILTON'S QUATERNION

In a 3-D Cartesian system a vector $\vec{R}$ will have projection cosines α, β, γ onto the three axes. If that vector represents a rotation in magnitude and axial direction, then the "quaternion" for that rotation is written

$$R = \cos\left(\frac{\theta}{2}\right) - i \sin\left(\frac{\theta}{2}\right)\left[\alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z\right]$$

where

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and θ is the rotation angle.

The three σ matrices have these properties:

$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$, $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$, and cyclic permutations.

If $\vec{V}$ is a space vector then the 2x2 matrix $X = V_x\sigma_x + V_y\sigma_y + V_z\sigma_z = \vec{V}\cdot\vec{\sigma}$ is transformed via

$$X' = R X \bar{R}$$

The new rotated space vector can then be reconstructed from

$$X' = \vec{V}'\cdot\sigma, \quad \text{where}$$

$$\vec{V}' = V_x'\,\hat{\imath} + V_y'\,\hat{\jmath} + V_z'\,\hat{k}.$$

The product R_3 of two rotations, R_1 followed by R_2, is $R_3 = R_2 R_1$.

Note that $\dfrac{dR}{dt} = \dfrac{-i}{2}(\vec{\sigma}\cdot\vec{\omega})R$, ~~and~~

where $\vec{\omega} = \omega\hat{m}$ is angular rotation rate in the sense of increasing θ.

This can be integrated to yield the rotation operator

$$R(\theta) = \exp\left(-i\left(\frac{\theta}{2}\right)\vec{\sigma}\cdot\hat{m}\right).$$

Errata and Comments on Lie Groups in Physics

provided by W. Daniel and D. Peterson,
entered by C. Kunasz (01/18/2023)

Errata:

pg. 8: The last sentence, beginning "In one useful gauge…", should be clarified, elaborated, and corrected to say: "The magnetic vector potential A is not uniquely determined. Because $B = \nabla \times A$ and $\nabla \times \nabla \phi = 0$, adding the gradient of a scalar function of A changes nothing observable. This is called "making a gauge choice". The Coulomb gauge ($\nabla \times A = 0$) is one such choice, but note that this does not imply $\phi = 0$. (W. Daniel). (See the **Comments** section below for more, offered by D. Peterson.)

pg. 11 (SU(2) section pg. 1)
Near the start of the SU(2) section. The first equation with SU(2) expressed as a matrix in alpha and beta, lists the (1,1) element as negative alpha-bar. This term should be positive. This can be verified by calculating the determinant which must be 1 as stated. The same problem appears in the the (1,1) term of the matrix just below in polar form. It is correct in the matrix that acts on the wave function superposition on page 12. (W. Daniel)

pg. 14 (SU(2) section pg. 4)
The page number is missing on this page. Also, in equation (5) near the bottom of the page, the argument of psi-sub-alpha on the left-hand side should be r-vector, not scalar. This is correct on the right-hand side. (W. Daniel)

pg 16 (SU(2) section pg. 6)
In equation (11) at the bottom of the page, the -i/h-bar J-sub-z on the right-hand side should be followed by a + U-sub-R of phi. This can be seen by explicitly working through the definition of

the derivative on on the left-hand side. It is also necessary for the conclusion on the top of the next page that the solution is an exponential. (W. Daniel)

Comments:

W. Daniel found the Lie Groups in Physics essay insightful and excellent overall. D. Peterson also liked it and said he "learned some things he didn't realize before". He appreciated equation (12) on page 17 and the fact that, lower down in that page the determinant of the exponential of A equals the exponent of the trace of A, so that since A is traceless, the determinant = exp(0)=1. He verified all of the errata points listed found by W. Daniel.

D. Peterson also commented that the motivation for a local change gauge is the restriction of light speed — quick global change defies relativity. Below are further helpful comments by Dave.

On pg. 6, Dave adds (after after the sentence mid-page about R(2) and R(3)),"Other conventional names for these groups are SO(2) and SO(3) for "special orthogonal groups in 2D and 3D with 'special' meaning determinant = +1".

On pg. 7, at the end of the definition of a compact group, Dave adds ", i.e. no domain punctures or missing end points."

On pg. 8, Dave was the first to note that the last sentence on pg. 8 was not okay. He pointed out, "We do not "have Φ=0". The mention of the Coulomb gauge *was* relevant as stated in the errata for pg. 8 above. Dave proceeded to provide an accurate, better, and detailed replacement as follows."We impose constraints on vector A such that $\nabla \bullet$ A=0 [Φ can still be non-zero]. The electric and magnetic fields (physical observables) can be derived from potentials according to Maxwell's equations for 'no poles', [i.e. $\nabla \cdot B=0$ allows for $B=\nabla \times A$] and Faraday's law [i.e. $\delta B/\delta t= -\nabla \times E$ is consistent with $-(\delta/\delta t)(\nabla \times A)=-\nabla \times(-\delta A/\delta t)= -\nabla \times E$ so that $E=-\delta A/\delta t$ and minus sign for law of induction]."

W. (Bill) Daniel noted that on pg. 16, the comment above eqn. (10) is beautiful, often mentioned, but the difficulty pointed out is less obvious and is relevant.

On pg. 18, Dave suggests an inserted comment at the beginning of III. HAMILTON'S QUATERNION, as follows. "**Hamilton's <u>quaternions are the generators</u>** of the Lie algebra SU(2) {basis 1, and complex ξ_1, ξ_2, ξ_3 — more often called i, j, and k}. These are related to the now more familiar Pauli spin matrices by: $\sigma_1=-i\xi_1$, $\sigma_2 = i\xi_2$, $\sigma_3 = i\xi_3$. Notice that the rotation matrix **R** uses combinations of imaginary i times the sigmas making them essentially quaternions. Pauli matrices are complex quaternions thus having a larger Clifford algebra than just quaternions. $Cl_{3,0}$ versus $Cl_{2,0}$ meaning 3 square roots of +1 and 2 square roots of -1 (only 2 needed since $i \cdot j = k$).

General comment from Dave:
Symmetry usually means symmetry of the Lagrangian. In classical physics, L=T-V=KE-PE, but the Schrodinger Lagrangian T-V also includes the ψ and ψ conjugate for the wave functions. The Schrodinger equation includes total T+V energy and ψ, so all the relevant energy parts and wavefunction are there, and we can discuss the symmetry of the Schrodinger equation.