So Long Dark Energy

Steven Crow
stevecrow314@gmail.com
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Abstract

The purpose of this note is to remedy an error in the usual approach to solving the Einstein field equations for an isotropic expanding universe. The error is a misinterpretation of the Friedmann equation as a first integral of the field equations, which are two second-order differential equations for the scale factor of the universe. Instead the Friedmann equation should be understood as a formula for the cosmological constant Λ that reconciles the two differential equations of the original field theory. The correct choice of Λ makes the two equations the same, a single second-order differential equation without Λ. With an appropriate choice for density, the solution for scale factor is the same as for the ΛCDM model of the cosmos. The current epoch of acceleration follows naturally from the Einstein field equations with no need for dark energy. The curvature k of the universe, moreover, is independent of density and can be set equal to its experimental value 0 with no need for density to have a special critical value.

Formulation

The analysis is based on the Einstein field equation for general relativity, including the cosmological constant. The field equation is a 4 by 4 symmetric tensor with ten independent terms. For an isotropic expanding universe, the off-diagonal terms are zero. The four diagonal terms include one for the time-time indices and three identical terms for the three dimensions of space. The result is two second-order differential equations for the cosmic scale factor $a(t)$:

$$\ddot{a} = -4\pi G\rho a/3 + \Lambda a/3\ , \tag{1}$$

$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G\rho a^2 + \Lambda a^2\ . \tag{2}$$

The symbols are defined as

$a(t)$	scale factor of the universe,
$\rho(t)$	density of the universe,
G	Newton's constant of gravitation,
k	curvature,
Λ	Einstein's cosmological constant, more recently called dark energy.

Equations (1) and (2) can be found as (15.1.18) and (15.1.19) in Weinberg's *Cosmology and Gravitation* [1], but without the cosmological constant. Equation (1) including the cosmological constant appears as (7.2) in [2], and an equivalent pair of equations appears as (10.9) in [3]. Space is "flat" or Euclidian when $k = 0$.

Both equations (1) and (2) are second-order differential equations for $a(t)$, and both can be solved in conjunction with an equation for conservation of mass,

$$\rho(t) = \rho_0 / a^3 , \tag{3}$$

and conditions for a and $\dot{a}$ at the current time t_0 :

$$a(t_0) = 1 , \qquad \dot{a}(t_0) = H_0 , \tag{4}$$

where H_0 is the current value of the Hubble factor and is measurable.

Alas the solutions of (1) and (2) would be different, because the equations themselves are different when regarded as equations for $\ddot{a}$. The way out of the dilemma is to find a value of the cosmological constant Λ that makes the two equations the same. That value proves to be

$$\Lambda = 3(\dot{a}^2 + k) / a^2 - 8\pi G\rho . \tag{5}$$

Equation (5) is usually presented with $\dot{a}^2$ on the left as the Friedmann equation [4] and treated as an almost magical first integral of the original second-order field equations (1) and (2). It is not, but that perception has given rise to various fallacies in relativistic cosmology, like the need for "dark energy" to accelerate expansion of the universe, and a coupling between k and ρ such that a critical value of density is needed to flatten the universe. The real function of (5) is to reconcile the field equations (1) and (2), which otherwise would yield different solutions $a(t)$

When (5) is inserted in (1) or (2), the result is a single second-order differential equation for scale factor:

$$\ddot{a} = (\dot{a}^2 + k)/a - 4\pi G\rho a . \tag{6}$$

Notice that the cosmological constant no longer appears in the equation. Equations (3), (4), and (6) are a complete prescription for an expanding isotropic universe. No "dark energy" appears, yet the equation replicates the results of the ΛCDM model of general relativity when density is assigned the same value, including visible and dark matter.

Recent observations of the cosmic microwave background indicate that $k = 0 \pm 0.005$ with 95% confidence [4], so k will be treated as zero for this analysis. That should not be surprising, since uniform density should produce no curvature. Thus

$$\ddot{a} = \dot{a}^2/a - 4\pi G \rho a \ . \tag{7}$$

The term $\dot{a}^2/a$ is accelerative and performs the role formerly assigned to dark energy. That term is not intuitive from the perspective of Newtonian mechanics, where a particle does not accelerate by virtue of its own kinetic energy. So be it. The Einstein field equations are not Newtonian mechanics.

Solution for Scale Factor

The purpose of this section is to solve the dynamics equation (7) for the scale factor $a(t)$. The solution is facilitated by introducing the time-dependent Hubble factor

$$H = \dot{a}/a \ . \tag{8}$$

Then

$$\dot{H} = \ddot{a}/a - \dot{a}^2/a^2 \ , \tag{9}$$

and (7) takes on the elegant form

$$\dot{H} = -4\pi G \rho \ , \tag{10}$$

which embodies all of general relativity for an expanding, isotropic, and Euclidian universe. When use is made of conservation of mass (3), equations (8) and (10) can be written together as

$$\dot{a} = Ha \ , \qquad \dot{H} = -4\pi G \rho_0 / a^3 \tag{11}$$

and integrated in tandem backward in time from the current conditions (4). The integration is a simple spreadsheet exercise.

An alternative is to find the first integral of (10) in the form $H(a)$, then integrate the first of (11) alone backward in time. That requires a bit more analysis, but $H(a)$ is needed for the distance-redshift formula in the next section. Start with

$$\dot{H} = (dH/da)(da/dt) = aH(dH/da) = a\left(dH^2/da\right)/2 \ . \tag{12}$$

The second member of (11) takes the form

$$dH^2/da = -\left(8\pi G \rho_0/3\right) da^{-3}/da \ , \tag{13}$$

which can be integrated backwards from the final conditions (4) to yield

$$H^2 = H_0^{\,2}\left[1+\Omega_0\left(1/a^3-1\right)\right] \; . \tag{14}$$

Ω_0 is the density parameter familiar from general relativity [2, pp 51-52],

$$\Omega_0 = 8\pi G\rho_0 \big/ \left(3H_0^{\,2}\right) \; . \tag{15}$$

Here Ω_0 should not be construed as a ratio of actual to critical density, because "critical density" is meaningless, as is "dark energy". Those concepts arise from misuse of the Friedmann equation.

Equation (14) provides the needed formula for $H(a)$, and the first of equations (11) can be written as

$$da/dt = aH_0\left[1+\Omega_0\left(1/a^3-1\right)\right]^{1/2} \; . \tag{16}$$

The relation between a and t can be found by a simple spreadsheet integration, and the result for $\Omega_0 = 0.22$ is shown in Figure 1. That value is taken from the data fit in the next section.

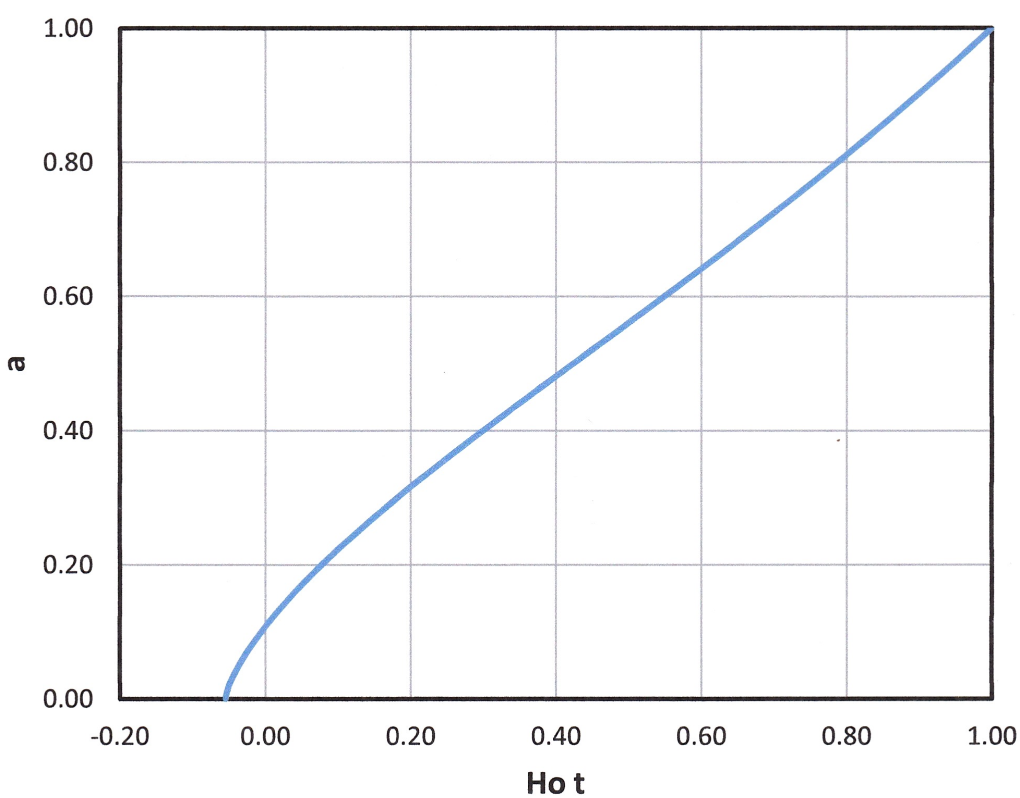

Figure 1. Scale factor a(t) for Omega0 = 0.22.

The curve is the same as would be obtained from the ΛCDM model of general relativity with $\Omega_0 = 0.22$ and $\Omega_\Lambda = 0.78$. The Big Bang occurs at $H_0 t = -0.055$, so the age of the universe is $1.055/H_0$. The transformation from deceleration to acceleration occurs at $H_0 t = 0.450$.

Distance-Redshift Relation

The function $a(t)$ offers an overview of the history of the universe, but neither a nor t is directly measurable. One quantity that is measurable is redshift z of light emitted from a distant source. If the scale factor was a_1 at the time t_1 of emission, then the redshift and scale factor satisfy the relation

$$a_1 = 1/(1+z) \quad . \tag{17}$$

[1, equation (14.3.6)]. Sometimes the source distance r_1 at the time of emission can also be measured independently of redshift, so the relation $r_1(z)$ can serve as an observable proxy for $a(t)$. To find that relation, we need a model for photon propagation from the source at time t_1 to ourselves at the current time t_0.

There are two ways to approach the model. The most common is to assume that light propagates along null geodesics in an expanding metric, like the FLRW metric of general relativity. That analysis can be found in [1, pages 415-416]. A second way is to assume that light propagates at a speed c relative to the local matter of the expanding universe. The two approaches give the same answer, but the second is more intuitive and is used here.

The equation for photon propagation is

$$\dot{s} = (\dot{a}/a)s - c \quad , \tag{18}$$

where $s(t)$ is the separation between the photon and the eventual observer. The sign of c is chosen so the photon is propagating toward the observer, as best it can against the stream of matter. At the beginning and end of the photon's journey,

$$s(t_1) = r_1, \quad \mathrm{s}(t_0) = 0 \ . \tag{19}$$

An integrating factor for (18) is $1/a$, so

$$d(s/a)/dt = -c/a \ . \tag{20}$$

By integrating from t_1 to t_0 and using (19), we find that

$$(r_1/a_1) = c\int_{t_1}^{t_0} dt/a(t) \;. \tag{21}$$

Equation (21) is the solution for the source distance when a is known as a function of t. The previous section gave $\dot{a}$ as a function of a, so a useful shortcut is to treat t as a function of a and substitute variables in the integral. Thus

$$dt = (dt/da)da = da/\dot{a}(a) \;, \tag{22}$$

and equation (21) becomes

$$(r_1/a_1) = c\int_{a_1}^{1} da/\left[a\dot{a}(a)\right] \;. \tag{23}$$

Replacing $\dot{a}$ with (16), we find that

$$r_1 H_0 \Big/ c = a_1 \int_{a_1}^{1} da \Big/ \left\{a^2\left[1+\Omega_0\left(1/a^3-1\right)\right]^{1/2}\right\} \;. \tag{24}$$

H_0 is the current value of the Hubble factor, Ω_0 is the density parameter (15), and r_1 is related to redshift z by (17). The ΛCDM model of general relativity produces the same formula as (24) when Ω_Λ is set equal to $(1-\Omega_0)$.

Figure 2 compares observations of distance versus redshift with the predictions of equation (24), with the right hand side expressed as a function of z. The data for z below 1.3 are taken from the supernova samples analyzed by Bertoule *et al*. [5]. The data point at $z = 7.085$ pertains to quasar ULAS J1120+0641 [6], and the extraordinary datum at $z = 1090$ is for the cosmic microwave background [7]. That datum has been interpolated leftward to $z = 1000$ to avoid having to add another decade to the log plot. The best eyeball fit of (14) to the data implies that

$$\Omega_0 = 0.22 \;, \qquad H_0 = 74 \text{ km/sec/Mpc} \;, \tag{25}$$

and the age of the universe is

$$1.055/H_0 = 13.94 \text{ Gy} \;. \tag{26}$$

The fit between equation (24) and the astronomical data appears excellent over five decades of redshift.

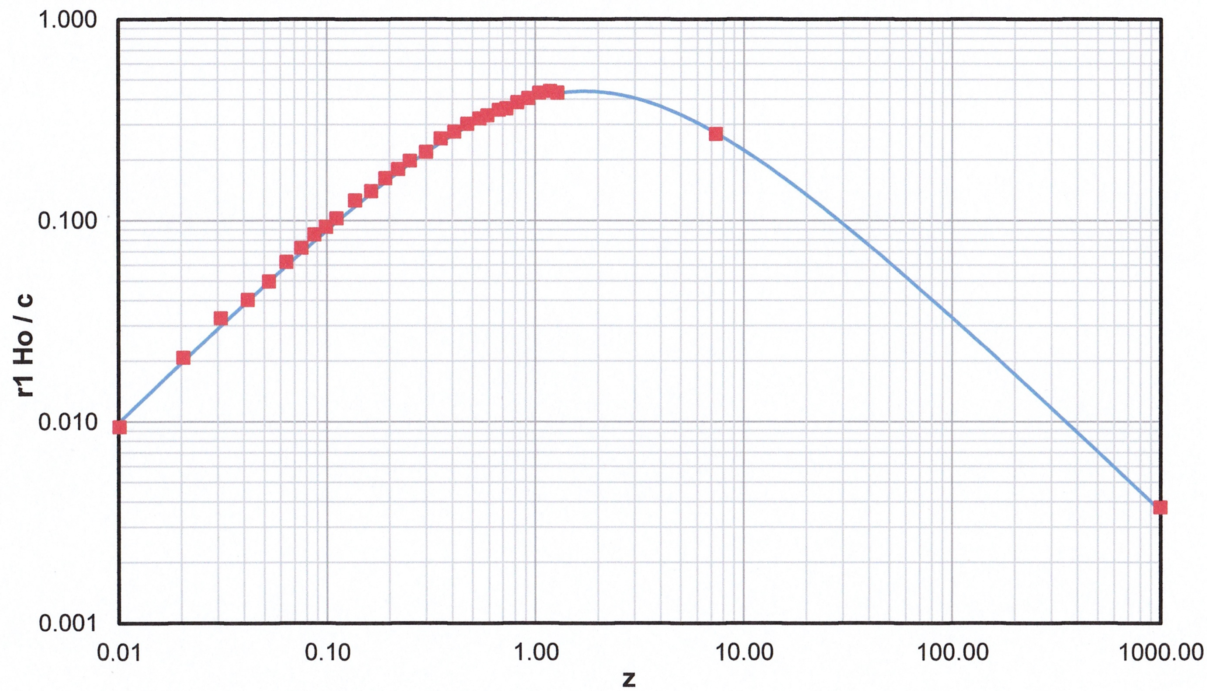

Figure 2. Distance-redshift relation for Omega0 = 0.22, H0 = 74 km/sec/Mpc.

Postscript

"To remedy an error" after 90 years of cosmological calculations may seem presumptuous, but the relatively simple mathematics speaks for itself. Recall that Einstein introduced the cosmological constant Λ to allow a universe of steady state but was happy to throw it out when Hubble discovered the expansion. Thereafter the field equations (1) and (2) were written for the most part without Λ, and the Λ-free version equation (5), the Friedmann equation, came to be regarded as a shortcut to the solution of the original second-order field equations (1) and (2). One result was an apparent need for the density of the universe to have a critical value to suppress curvature to its observed value of zero. Another was the need for "dark energy" to provide an impetus for the accelerated expansion observed in the 1980's. With the correction proposed here, curvature can be zero regardless of density, and acceleration emerges as a natural consequence of the Einstein field equations without need for dark energy.

References

1. Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley and Sons, 1972.

2. Liddle, A. *An Introduction to Modern Cosmology, Third Edition*. John Wiley & Sons, 2015.

3. Bondi, H. *Cosmology.* Cambridge University Press, reprinted 1961.

4. Friedmann, A. *Z. Phys*. **21**, page 326, 1924.

5. Betoule et al. Improved cosmological constraints from a joint analysis of the SDSS-II and SNLS supernova samples. *Astronomy & Astrophysics*, 8 August 2014.

6. Wikipedia ULAS J1120+0641. **https://en.wikipedia.org/wiki/ULAS_J1120%2B0641**, 2013.

7. Planck Collaboration *Planck* 2015 results. XIII. Cosmological parameters, 5 February 2015.